Theoretical calculation of wind (Or water) turbine considering kinetic and potential energy to exceed the Betz limit

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The Betz limit sets a theoretical upper limit for the energy efficiency of turbines with a maximum power coefficient of 16/27. Betz's theory is precise and is based on the calculation of kinetic energy. However, if the potential energy is taken into account the theoretical energy efficiency of a turbine can be higher. Fast wind turbines recover the kinetic energy of the wind in an optimal way. A large amount of potential energy is created without being recovered. The notion of potential energy is fundamental, it is not possible to recover energy, if we do not create a stress. We examine this potential energy and the possibility for a wind turbine to transform it into kinetic energy. The Betz theory has been defined from the model of fast moving turbines. This theory has been generalized to slow and fast moving turbines and it has been defined as a law. The conservation of energy implies that if a variation of kinetic energy increases, the variation of potential energy decreases. In the case of slow moving turbines, the conservation of energy applies, but not for the case of fast moving turbines, however this is the reality.
The paper proposes a new formulation of the turbine power with a notion of temporal, in order to be able to verify the conservation of energy.

Keywords Betz-limit Betz's-law Wind-turbine Tidal-turbine HAWT VAWT ALWT

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1 Nomenclature

This list is an enumeration of the main symbols used in this article.

Symbol	Designation	Unit
Roman symbol		
V_{fluid}	Wind velocity	$\frac{m}{s}$
b	Plenitude speed ratio ($b = \frac{\dot{\beta}Nc}{V_{fluid}}$ $b \leq \frac{4}{3}$)	–
c	Chord length of blades	m
$C_{kin\ Betz}$	Maximum kinetic power coefficient ($C_{kinBetz} = \frac{16}{27}$)	–
C_{kin}	Kinetic power coefficient	–
C_{pot}	Potential power coefficient	–
C_{total} or C_T	Total power coefficient	–
e	eccentric distance	m
E_{kin}	Kinetic energy	J
E_{pot}	Potential energy	J
E_{total}	Total energy ($E_{total} = E_{kin} + E_{pot}$)	J
F_a	Axial force (axial to the blade profile)	N
F_n	Normal force (normal to the blade profile)	N
H	Blade height	m
N	Number of blade	–
P_{fluid}	Fluid power ($P_{Fluid} = \frac{1}{2}\rho S_{fluid}V_{fluid}^3$)	W
P_{kin}	Kinetic power	W
P_{pot}	Potential power	W
S	Swept area	m^2
S_{fluid}	upstream stream-tube cross-sectionnal area $S_{fluid} = \frac{S}{a}$	m^2
U	Rotation speed	$\frac{m}{s}$
W	Relative speed	$\frac{m}{s}$
a	Velocity factor $a = \frac{V}{V_{fluid}}$	–
V	Fluid velocity at the position of the turbine	$\frac{m}{s}$
Greek symbol		
β	Rotation angle of the blades	rad
λ	Tip ratio ($\lambda = \frac{\omega\ R}{V_{fluid}}$)	–
ρ	Fluid density	$\frac{kg}{m^3}$
σ	Stress in turbine blade	$\frac{N}{m^2}$
Abbrevations		
$ALWT$	Active lift wind turbine	–
$HAWT$	Horizontal-axis wind turbine	–
$VAWT$	Vertical-axis wind turbine	–

2 Introduction

Lanchester, Betz, Joukowsky have defined the maximum power coefficient of wind turbines (Van Kuik 2007). This limit is commonly called the Betz limit. Signifiant research efforts have been deployed to optimize wind turbines in order to reach this limit, for instance by optimizing the angle of incidence, the shape of the blade profile etc. One may for example refer to "Wind Energy Handbook" Burton et al. for fast moving horizontal axis wind turbines (HAWT) (Burton et al. 2006), or to "Hydrodynamic modelling of marine renewable energy devices: A state of the art review" Day et al. (2015) (Day et al. 2015) or to "Wind Turbine Design: With Emphasis on Darrieus Concept" for vertical axis wind turbines (VAWT) (Paraschivoiu n.d.), or to "Wind turbines: theory, design and practical calculation" (Le Gourieres 2008).
Research has been made to reach the Betz limit:
– by ducting the turbine(Georgiou et al. 2016)
– by placing a water turbine in a narrow channel, which allows the water level upstream to increase due to the resistance of the advancing fluid (Quaranta 2018)

– by grouping turbines in wind farms in order to create an excess of power due to the proximity between machines (ducting effect)(Vennell 2012) (Broberg et al. 2018).
Studies are still being carried out to try to obtain maximum gain and exceed the Betz limit, for example : (Aravindhan et al. 2023), (Hesami et al. 2023), (Quispe-Abad et al. 2023), (Pellegri 2022), (Dabiri 2020), (Murthy et al. 2017), (Marzec et al. 2021), (Barnes et al. 2021).
An interesting study for exceeding the Betz limit is that of "Peter A. Sharp"(Sharp 2021).
. The Betz limit is based on the calculation of kinetic energy. Designing a wind turbine, the energy of the fluid is taken into account in order to calculate the recoverable energy and to design a structure withstanding the stress. There is a contradictory difference between the stresses on a square wing figure 1 and on a modern wing figure 2. Betz's limit is based solely on wind turbines with wing profiles. In the case of a sailboat, the sole sail cannot recover all the energy from the wind. Betz's theory applies to the sail that can not retrieve more than 16/27 of the wind's kinetic energy. Many books explore the Betz limit, for instance John Kimball's "Physics of Sailing" (Kimball 2009). However, the America's Cup is a good example showing that Betz's limit only takes kinetic energy into account. Indeed, a sailboat equipped with hydrofoils transforms potential energy into kinetic energy. The kinetic energy of the wind sets the boat in motion. The fluid flowing around the hydrofoils lifts the boat out of the water, thus reducing the drag and improving the efficiency. The wind power has not increased but the potential energy that applied stresses on the keel has been converted into kinetic energy by the flow of fluid around the foils profiles. This article introduces the notions of kinetic and potential energy. It suggests that a vertical axis wind turbine (VAWT) with an appropriate design can transform potential energy into kinetic energy.

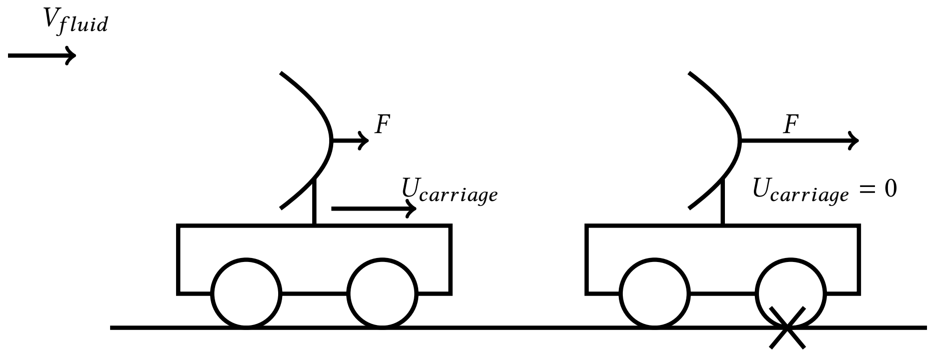

Figure 1: When a carriage with a square sail is stationary, in the presence of wind, the forces on the sail are high and decrease as soon as the carriage is in motion.

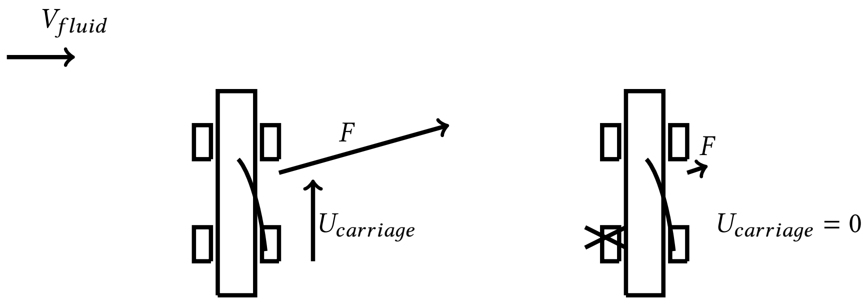

Figure 2: Unlike the case of the carriage with a square sail figure 1, the forces on a modern sail are low when the carriage is stationary and increase as soon as the carriage is in motion.

3 Preliminary considerations of Betz's theory

The German mathematician A. Betz has demonstrated that the power of a turbine is :

$$P_{kin} = F\,V \;= [\,\frac{C_{kin}}{a}\,\frac{1}{2}\,\rho\,S\,V_{fluid}^2]\;a\,V_{fluid} = C_{kin}\,\frac{1}{2}\,\rho\,S\,V_{fluid}^3 \tag{1}$$

with differents symbols specified in the nomenclature. According to Betz 's work (preliminarily by (**Lanchester**), a kinetic energy approach shows that the maximum power coefficient C_k can not exceed a maximum of :

$$C_{kin\ Betz} = \frac{16}{27} \tag{2}$$

figure 3 shows the forces applied to a HAWT turbine. The relative speed due to the rotation speed of the wind turbine and the fluid speed creates an induced force F on the profile. This induced force F is composed of an axial force F_a and of a normal force F_n. The axial force F_a associated with the radius R creates a driving torque to produce energy. The normal force F_n associated with the same radius R creates a bending stress on the blades. The power of the torque due to the axial

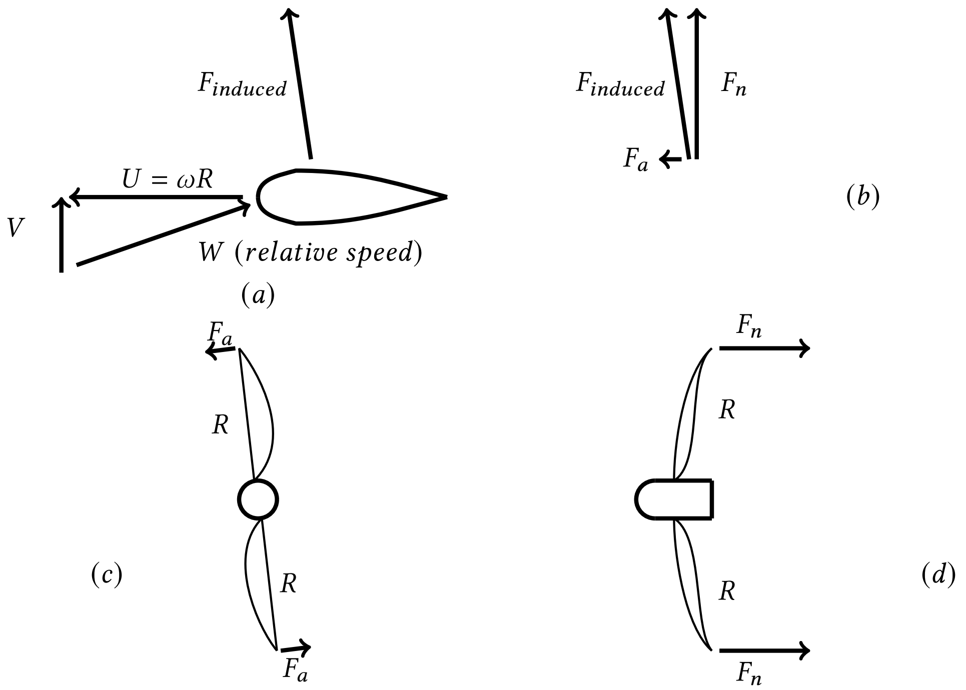

Figure 3: Forces applied to a HAWT turbine.
(a) *The relative speed creates an induced force on the profile*
(b) *Decomposition of the induced force*
(c) *Driving torque*
(d) *Bending stress in the blades*

forces and the angular rotation speed of the wind turbine can not exceed 16/27 (equation (2)) of the wind's kinetic power. The forces Fn and Fa are associated with the same radius and have the same origin: the fluid velocity. Fn creates stresses that are the source of internal energy. These stresses are potential energy. Betz's theory does not take into account this potential energy which is as important as the kinetic energy. So the Betz limit is consistent. However, it only takes into account kinetic energy. In order to increase the efficiency of wind and hydrokinetic turbines, they could be designed to convert the potential energy into kinetic energy. It is necessary to dissociate slow speed turbines and fast-speed turbines. in the case of fast-speed turbines, the tangential rotation speed is higher than the fluid speed, the λ coefficient is a multiplier defined by

$$\lambda = \frac{U}{V_{fluid}} = \frac{\omega R}{V_{fluid}} \tag{3}$$

λ is an important parameter which makes a difference in the behaviour of the turbines when exposed to wind. As for a sail, it is necessary to dissociate the navigation in thrust (square sail, spinnaker : unstuck flow) and navigation in smoothness (wing profile sail : laminar flow).

4 Difference between a low and high speed turbine

4.1 The advantage of using an airplane wing

Fast moving turbines can not be described in the same way as slow moving turbines such as Savionus. The comparison is made between the airfoil of a fast moving turbine and the cup of an anemometer in Figure 4. If the movement of an anemometer is not impeded, then its cups are

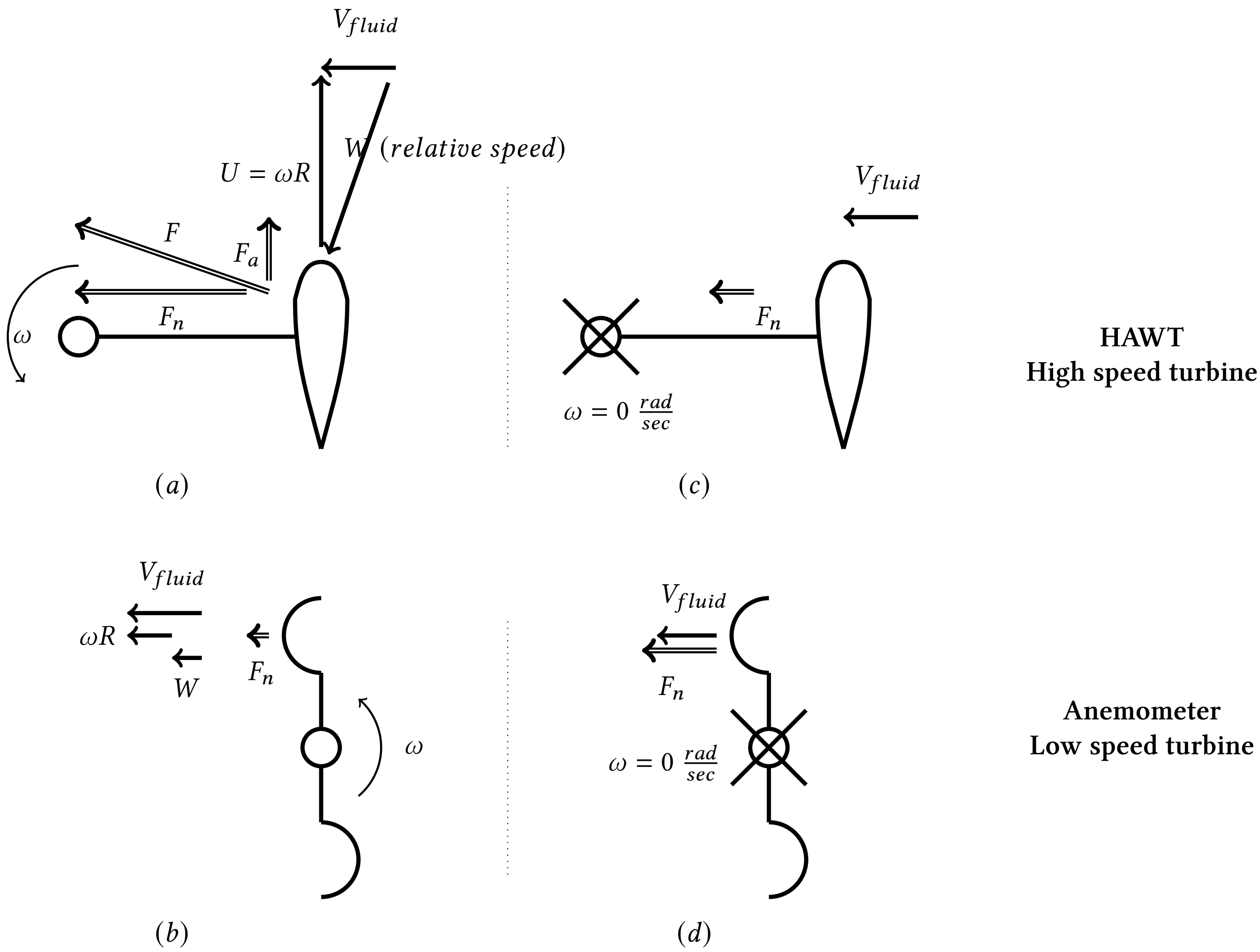

Figure 4: Comparaison between HAWT (a)-(c) and Anemometer (b)-(d)
(c) and (d) are the cases blocked in rotation.

almost free of stress. If the anemometer rotation is hampered, then its cups are subject to stress. In the case of a wing profile, the result is totally different. Indeed, the tangential rotation speed U of the profile is transverse to the fluid flow. The relative speed W due to the rotation speed U and the fluid speed V_{fluid} creates an induced force F on the wing profile. The axial component F_a of this force F, combined with the radius, creates a driving torque.
For *horizontal axis wind turbines* (HAWT) (figure 4 & table 1b), this torque improves the efficiency which can then approach the Betz limit. Adding a transverse speed U will optimize the efficiency defined by Betz, but will add significant stress on the wind turbine due to the normal component F_n of force F. That's why HAWT wind turbines have to be stopped when the speed of the fluid is too high. They do not produce too much but they are subject to excessive bending stress.
The appendix A.1 and table 1 present energy variations. In the case of an anemometer(figure 4 & table 1a), the kinetic energy varies in a direction opposite to the potential energy. This is not the case for HAWT turbines(figure 4 & table 1b). In this case, we must not only consider the fluid speed. The kinetic energy and potential energy are related to the fluid speed and to the tangential rotation speed U. The direction of U is perpendicular to the direction of the fluid. As λ is an important parameter, it is dealt with in terms of energy in the following paragraphs with consider low-speed and high-speed turbines.

a : Anemometer Low speed turbines ($\lambda < 1$)

case	$U = \omega R$	W	F_n	$Stress$	$torque$	$potential\ energy$	$kinetic\ energy$
$blocked$	0	$= V_{fluid}$	max	max	nul	max	$null$
$free$	$\nearrow$	$\searrow$	$\searrow$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

As soon as the turbine is free, the stresses decrease and the torque increases.

b : HAWT High speed turbines ($\lambda > 1$)

case	$U = \omega R$	W	F_n	$Stress$	$torque$	$potential\ energy$	$kinetic\ energy$
$blocked$	0	$= V_{fluid}$	min	min	$null$	min	$null$
$free$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$	$\nearrow$

As soon as the turbine is free, the stresses increase and the torque increases.

Table 1: Energy variations between the blocked and rotating turbine situations for a anemometer (a) and a HAWT (b)

4.1.1 Low speed turbines $\lambda < 1$

Using the example of an anemometer with cups, when there is no resisting torque, the turbine rotates at maximum speed ($\omega R = V_{fluid}$ and $W = 0$), there is no energy production and the kinetic energy is at its maximum. When the turbine is blocked, the stress on the turbine is at maximum ($W = V_{fluid}$) and the rotation speed is null. The potential energy is at its maximum. To obtain energy production, both kinetic and potential energy are required. If the kinetic energy is higher than the potential energy, the energy production is limited by the potential energy. Similarly, if the potential energy is higher than the kinetic energy, the energy production is limited by this latter.

$$E_{kin-max-productive} \leq min(\ E_{kin}\ ,\ E_{pot}\ ) \tag{4}$$

There can be no production of kinetic energy without potential energy. If the kinetic energy increases, the potential energy decreases and vice versa :

$$\frac{dE_{kin}}{dt} \nearrow \frac{dE_{pot}}{dt} \searrow \qquad or \qquad \frac{dE_{kin}}{dt} \searrow \frac{dE_{pot}}{dt} \nearrow \tag{5}$$

Energy production is at its maximum when the kinetic energy is equal to the potential energy. The total energy is equal to the sum of the kinetic energy and the potential energy.

$$E_{total} = E_{kin} + E_{pot}$$

The conservation of energy (equation (18) of appendix A.1) can be applied and verified.

$$\frac{dE_{total}}{dt} = 0 \qquad \longrightarrow \qquad \frac{dE_{kin}}{dt} = -\frac{dE_{pot}}{dt} \tag{6}$$

For a low speed, the kinetic energy of the wind is partially transformed into kinetic and potential energy.

$$E_{k\,wind} \Rightarrow E_{pot}\ \&\ E_{kin} \tag{7}$$

4.1.2 High speed turbines $\lambda > 1$

Using the example of a turbine with wing profile blades, the variation of kinetic and potential energy increases simultaneously or decreases simultaneously.

$$\frac{dE_{kin}}{dt} \nearrow \frac{dE_{pot}}{dt} \nearrow \qquad or \qquad \frac{dE_{kin}}{dt} \searrow \frac{dE_{pot}}{dt} \searrow \tag{8}$$

Thus, if we apply the conservation of energy principle $\frac{dE_{kin}}{dt} = -\frac{dE_{pot}}{dt}$, the equation is not verified, even though it is reality. Justifying that kinetic energy and potential energy increase simultaneously is possible because the lambda tip ratio is greater than 1, is not a sufficient reason. Obtaining a velocity coefficient greater than 1 can only be achieved by adding energy. An additional parameter must be taken into account to verify simultaneous energy increases.

4.2 Consideration of a additional parameter

The change of direction of the fluid (figure 5) is due to the stresses on the blades. These induce a change of direction of the fluid and create a driving torque. The fluid is in motion and has kinetic energy. The fluid in contact with the turbine exerts a pressure field and creates stresses in it. These stresses allow the fluid to change direction and gives the possibility to create a driving torque.

These three successive phenomena can be interpreted on temporal steps :

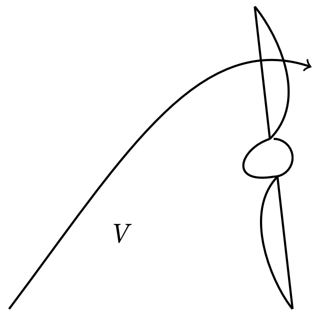

Figure 5: Change in fluid direction due to interaction with the blades

The fluid ① creates stress on the surface swept ② by the turbine, then there is a change of direction ③ of the fluid .
① Kinetic energy of the fluid
② Stresses are created on the turbine
③ The change of direction of the fluid creates a driving torque to generate energy
The figure 6 is a visual schematic of these three steps : ①, ② and ③.
Successive phenomena involve a notion of time.

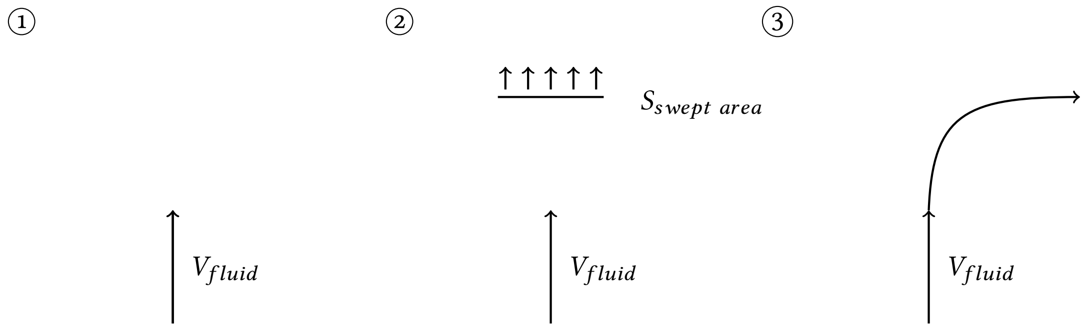

Figure 6: Symbolization of the three energy transfer stages

In terms of energy, there is a transfer of temporal energy. There is a transformation of energy from the kinetic energy of the fluid into potential energy, then into kinetic energy and the energy can only decrease in the streamwise.

$$E_{kwind} \Rightarrow E_{pot} \Rightarrow E_{kin} \qquad E_{kwind} \geq E_{pot} \geq E_{kin} \tag{9}$$

The figure 7 shows the energy transfers, taking instantaneous times into account. As the transfer of energy is not simultaneous, $E_{kwind}(t) \Rightarrow E_{pot}(t+\Delta\ t) \Rightarrow E_{kin}(t+2\Delta\ t)$. Simultaneously, we have three energies : $E_{kwind}(t),\ \ E_{pot}(t-\Delta\ t),\ \ E_{kin}(t-2\Delta\ t)$. In the case of an increase in the speed of the fluid, we have successively the following energy variations

$$\frac{dE_{kwind}(t)}{dt} > 0 \quad \Rightarrow \quad \frac{dE_{pot}(t+\Delta\ t)}{dt} > 0 \quad \Rightarrow \quad \frac{dE_{kin}(t+2\Delta\ t)}{dt} > 0 \tag{10}$$

In the paragraph 4.1, it was shown that there was a lack of respect for the energy conservation. By introducing the temporal concept, the formulation equation (10) allows to be consistent with the energy conservation. For a sand yacht, if you set the sail directly to the right position, it will

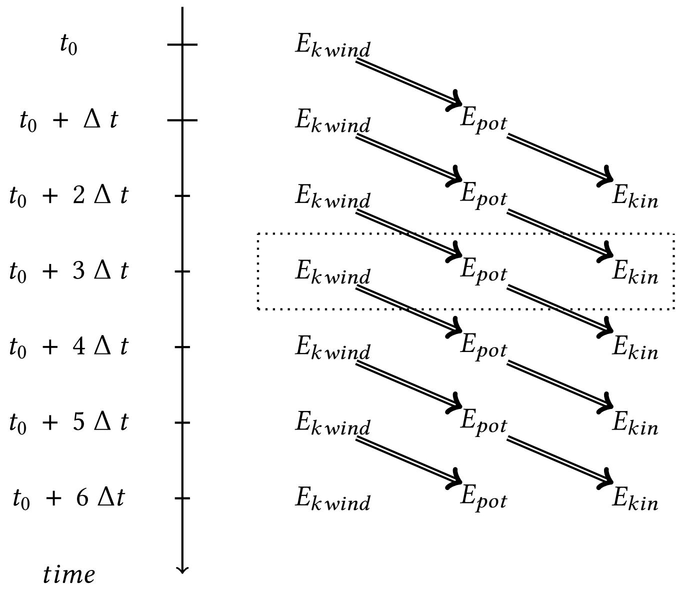

Figure 7: Energy transfert with temporal steps

not move forward or it will take a long time to reach its optimum speed. If you tuck in the sail little by little to put it in the right position, the sand yacht has the right speed. When stopped, the sail with the sail directly in the right position undergoes pressure forces due to the wind and this one is forced to go around the obstacle which is the sail. By gradually putting the sail in the right position, the tank picks up speed, the wind changes direction. When stopped with the sail in the right position, the wind creates stresses on the sail and creates a force that tries to tilt. By gradually putting the sail in the right position, the wind combined with the forward speed, creates an induced force which allows the sailboat to move forward and which also creates a tilting force. In fact, if there is no stage step ② (stresses on the sail to create an induced force which allows the sand yacht to move forward), stage step ③ which allows the fluid to change direction does not perform. The temporal concept is important.
In steady state, the induced force allows the sailboat to move forward (step ②) and then allows the fluid to change direction (step ③). As step ③ is done,the step ② is maintained. We have a steady state of operation.

4.3 Power coefficients of a turbine

The paper of (Raghed n.d.) presents the calculation of Albert Betz who determined the power coefficient from kinetic energy. He uses that power is the change in energy relative to the change in time. To determine the force acting on the swept surface of the turbine, he uses Euler's theorem. By multiplying the force by the fluid speed at the turbine, he again obtains a power term. From these power equations, it determines the speed of the fluid at the turbine and determines the maximum power coefficient which is called the Betz coefficient. In fact, to determine the force power exerted by the turbine rotor, it uses the variation of potential energy and the Euler equation. Power can be expressed as follows $P_{index} = C_{index}\,\frac{1}{2}\,\rho\,S\,V_{fluid}^3$. Power is the change in energy relative to the change in time. In the previous paragraph, the notion of time was introduced for energy variations (equation (10)). As the power coefficient of the fluid is equal to $C_{fluid} = 1$ and the power coefficient due to kinetic energy is $C_{kin} = 4a^2(1-a)$, we have this inequality

$$C_{fluid} \geq C_{pot} \geq C_{kin} \qquad\qquad 1 \geq C_{pot} \geq 4a^2(1-a) \tag{11}$$

This inequality is consistent with references equation (9) and equation (8).

5 Conversion of potential energy into additional kinetic energy

There can be energy production only if we have both kinetic and potential energy.

5.1 Conversion of potential energy

It is possible to convert potential energy into kinetic energy. In the case of horizontal wind turbines (HAWT, fast wind turbine type), the stress in the blades for a defined wind speed is constant.

$$\frac{d\sigma}{d\beta} = 0 \tag{12}$$

With σ Stress in turbine blade and β rotation angle of the blades. In fact, some stress variations exist due to gravitational forces and the differencial velocity within the boundary layer depending on the elevation. In the case of vertical axis turbines (VAWT, Darrieus type) the blades stress and arms depends on the rotation angle of the blades (for a given wind speed).

$$\frac{d\sigma}{d\beta} \neq 0 \qquad \frac{1}{2\,\pi}\int_0^{2\,\pi} \sigma d\beta \;=\; \epsilon \;\; (\epsilon\ small) \tag{13}$$

During a half-turn, the arms are submitted to compression stress whereas extending stress is dominant during the next half-turn. In the case of a HAWT, the conversion is not possible with a dynamic mechanical system. The additional stress is constant during a rotation for a given wind speed. Alternative stress, encountered in a vertical axis wind turbine VAWT can allow the extraction of additional energy. In the case of a VAWT turbine, the conversion of potential energy into kinetic energy can only be achieved if there is radial displacement of the blade. The power obtained depends on the normal force and the radial displacement speed. Since the normal force creates compressive stress during one half-turn and tensile stress during the other half-turn, the radial displacement speed is related to the turbine's rotational speed. The radial speed depends on the variation in the blade's radius. The variation in radius must be significantly less than the radius. It is impossible to achieve total conversion of potential energy into mechanical energy. A conversion of 20 or 30% is feasible.

5.2 Total recoverable power of the turbine

We select the power defined from the kinetic energy and in the case of conversion of stress into a mechanical movement, the power defined from the potential energy. In this case, it is possible to convert potential energy into kinetic energy.
The total recoverable power coefficient is

$$C_{total} \;=\; C_{kin} \;+\; C_{pot} \tag{14}$$

With $C_{pot} = 0$ when stresses are not converted, the fuid power is

$$P_{total} = \; C_{total}\,\frac{1}{2}\rho S V_{fluid}^3 \;=\; C_{kin}\,\frac{1}{2}\rho S V_{fluid}^3 \tag{15}$$

In the case of horizontal wind turbines (HAWT, fast wind turbine type), the power coefficient is (reference equation (23))

$$C_{Total\ HAWT} \;=\; C_{kin} \;=\; 4\,a^2\,(1-a) \quad with \quad C_{pot} = 0$$

In the case of vertical axis wind turbines (VAWT, Darrieus type), the power coefficient is

$$C_{Total\ VAWT\ Darrieus} \;=\; C_{kin} \;=\; 4\,a^2\,(1-a) \quad with \quad C_{pot} = 0$$

In the case of vertical axis wind turbines (VAWT with conversion), this stress is converted into additional energy. Referring to equation (11), the power coefficient C_{pot} is greater than the power coefficient C_{kin} The power coefficient is therefore equal to

$$C_{Total\ VAWT\ with\ conversion} \;=\; C_{kin} \;+\; C_{pot} \;\geq\; 4\,a^2\,(1-a)$$

a : First case - optimum for high-speed turbines

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
in theory	C_{kin}	60%	60%	60%
in theory	C_{pot} *w.c*	0%	0%	60%
in theory	C_{total}	= 60%	= 60%	= 120%
in practice	C_{kin}	$0.8 \times 60\% \approx 48\%$	$0.7 \times 60\% \approx 42\%$	$0.7 \times 60\% \approx 42\%$
in practice	C_{pot} *w.c*	0%	0%	$0.6 \times 0.7 \times 60\% \approx 25\%$
in practice	C_{total}	= 48%	= 42%	= 67%
....				
gain/HAWT		+ 0%	−12 %	+ 39%
gain/$C_{kin}(Betz)$		− 20%	−30 %	+ 11%
$With\ a = \frac{2}{3}$	$C_{kin} \approx 60\%$	$C_{pot} \approx 60\%$	(*equation* (24) & *equation* (25))	

b : Second case - optimum for slow-speed turbines

case	Coef.	HAWT	VAWT Darrieus	VAWT with conversion
in practice	C_{kin}	$0.8 \times 50\% \approx 40\%$	$0.7 \times 50\% \approx 35\%$	$0.7 \times 50\% \approx 35\%$
in practice	C_{pot} *w.c*	0%	0%	$0.6 \times 0.7 \times 50\% \approx 21\%$
in practice	C_{total}	= 40%	= 35%	= 56%
......				
gain / HAWT		+ 0%	−12 %	+ 27%
gain / C_k		− 20%	−30 %	+ 12%
$With\ a \approx 0.8$	$C_{kin} \approx 50\%$	$C_{pot} \approx 50\%$	(*equation* (21) & *equation* (22))	

Table 2: Performance comparison (notation $C_{pot}\ w.c = C_{pot}\ with\ conversion$)

6 HAWT-VAWT comparison

Following the work of (Hau 2000), the power coefficient of different turbines is compared (a performance of 0.6 is applied for the supplementary energy recovery system). The performance comparison (table 2) is made in 2 cases. The first when the optimum is obtained for high-speed turbines ($\lambda > 1$) and the second case for slow-speed turbines ($\lambda < 1$).
The figure 8 and table 3 are a graphical representation and expression of the power coefficients as a function of velocity factor a.

Curve number	Coefficient name	Mathematic function
5	$C_{T\ Maxi}$	$1.0\ (C_{kin} + C_{pot})$
4	$C_{T\ VAWT\ with\ conversion}$	$0.7\ (\ C_{kin}\ +\ 0.6\ C_{pot})$
3	$C_{kin\ Betz}$	$1.0\ C_{kin}$
2	$C_{T\ HAWT}$	$0.8\ C_{kin}$
1	$C_{T\ VAWT\ Darrieus}$	$0.7\ C_{kin}$
	C_{kin}	$4\ a^2\ (1\ -\ a)$
	C_{pot}	$4\ a^2\ (1\ -\ a)$

Table 3: Expression of power coefficient HAWT VAWT (representation figure 8)

6.1 The Active Lift Wind Turbine (ALWT)

The Active Lift Wind Turbine (ALWT) is a horizontal turbine (HAWT) with a crank and connecting rod system that converts the normal forces on the airfoils into additional energy-recovery torque. The "Active lift wind turbine" is an example of transformation of potential energy into kinetic energy (**lecanu:hal-01300531**). Figure 9 shows a graphical representation of the "rod and crank" conversion system.

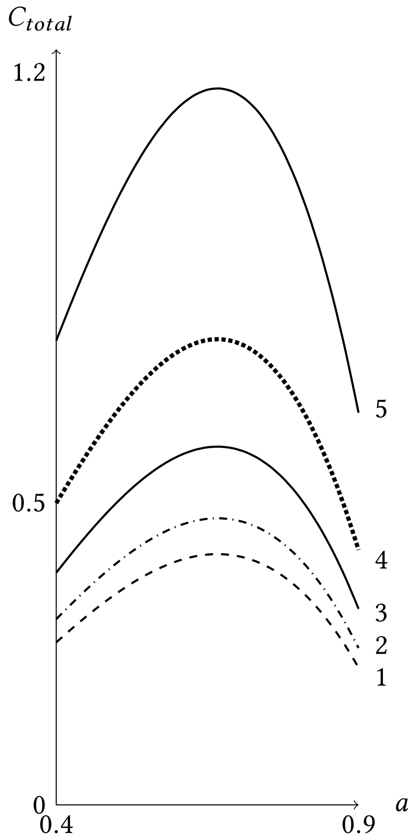

Figure 8: Representation of power coefficient HAWT VAWT (expression of coefficients table 3)

For this turbine, the power coefficient is equal to

$$C_{active.lift.turbine} = (1 + \frac{e}{R}\lambda)(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)$$

$$with \quad b = \frac{\dot{\beta}Nc}{V_{fluid}} \qquad \lambda = \frac{R\dot{\beta}}{V_{fluid}} \qquad b \leq \frac{4}{3} \qquad b: \ Plenitude\ speed\ ratio$$

NB: When e =0, the power coefficient corresponds to that of a Darrieus turbine $C_{kinDarrieus} < \frac{16}{27}$. The optmization of the maximum coefficient $C_{active.lift.turbine}$ with respect to the coefficient b is obtained by

$$\frac{dC_{active.lift.turbine}}{db} = \frac{d[(1 + \frac{e}{R}\lambda)(\frac{9\pi}{2^7}b^3 - \frac{2^2}{3}b^2 + \frac{\pi}{2}b)]}{db} = 0 \qquad (16)$$

$$if\ e = 0.2 \qquad b_1 \approx 0.84 \qquad C_{active.lift.turbine} \approx .85 \qquad (17)$$

Figure 10 shows a graphical representation of the turbine power coefficient.

7 Analysis between low and high speed turbine

A graphical diagram shows the difference between low- and high-speed turbine energy processing (respectively figure 11 and figure 12).

7.1 Low speed turbine $\lambda < 1$:

In the case $\lambda < 1$ (turbine type Anemometer with cups), at the energy level, we have with the equation (7) $E_{kin\ wind} \Rightarrow E_{pot}$ & E_{kin}.
As fluid velocity increases, we simultaneously have these energy variations (ref. equation (5)):

$$\frac{dE_{pot}}{dt} < 0 \quad \& \quad \frac{dE_{kin}}{dt} > 0$$

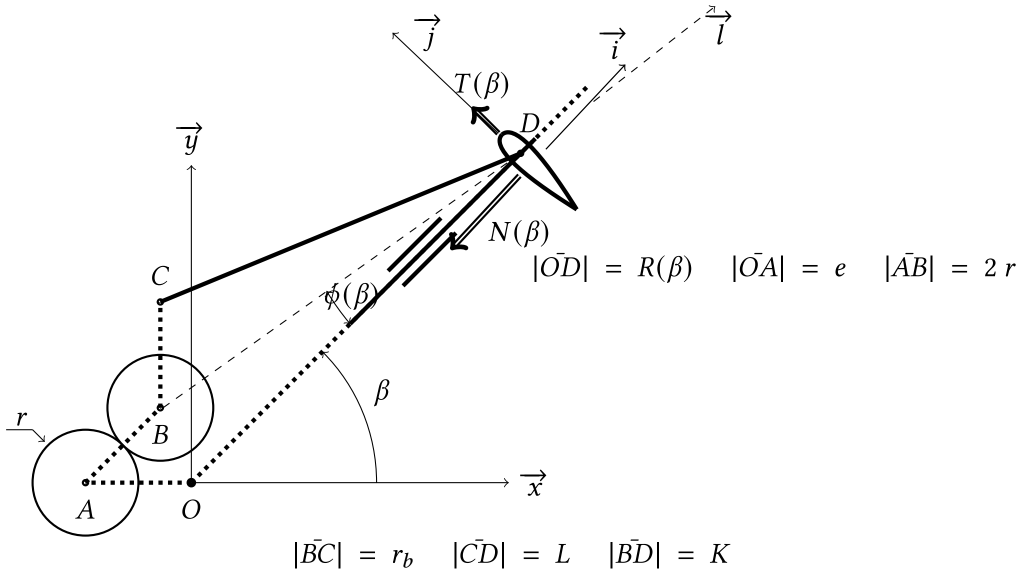

Figure 9: Schematization of the crank-rod system of ALWT

The figure 11 a is a diagram at energy level. The energy of the fluid is converted partly into potential energy and partly into kinetic energy.

7.2 Fast speed wind turbines HAWT or VAWT $\lambda > 1$:

In the case $\lambda > 1$ (large wind turbines HAWT or VAWT) , there is a temporal energy transfer equation (9): $E_{kwind} \Rightarrow E_{pot} \Rightarrow E_{kin}$. If the fluid velocity increases, there is an increase in stress on the blades and a blade rotation speed increase. This translates into the following energy variations (ref. equation (8)).

$$\frac{dE_{pot}}{dt} > 0 \quad \Rightarrow \quad \frac{dE_{kin}}{dt} > 0$$

The figure 12 is a diagram at energy level. In the case of a turbine with a lambda lower than 1, a part of the recoverable energy, is transformed into kinetic energy and the other part into potential energy (stress in the turbine). A part of this potential energy can be transformed into kinetic energy. In the case of a turbine with a lambda higher than 1, the recoverable energy defined by Betz, is transformed into potential energy. This energy creates kinetic but also potential energy (stress in the turbine). This is possible because a rotation speed which is higher than the speed of the fluid is created. A part of this potential energy (due to the stress) can be transformed into kinetic energy.

7.3 Summary of the coefficients of the maximum extracted powers:

In order to obtain productive kinetic energy, kinetic energy and potential energy are needed. The maximum productive kinetic power is equal to (ref. equation (4))

$$P_{k-max-productive} \leq min(\; P_k \; , \; P_p \;)$$

The limit of this depends on whether it is a low or high speed turbine:

$$P_{kin(max)} = C_{kin(max)} \, \frac{1}{2} \rho S V_{fluid}^3$$

Where $C_{kin(max)} = \frac{1}{2}$ for low speed turbines ($\lambda < 1$) (ref. equation (22)) and $C_{kin(max)} = \frac{16}{27}$ for high speed turbines ($\lambda > 1$) (ref. equation (25)).
In reference to the equation (11), with total conversion of potential energy into kinetic energy, the maximum power coefficient for a wind or tidal turbine is

$$1 \geq C_{Total\ maxi\ \lambda<1} \geq \frac{1}{2}$$

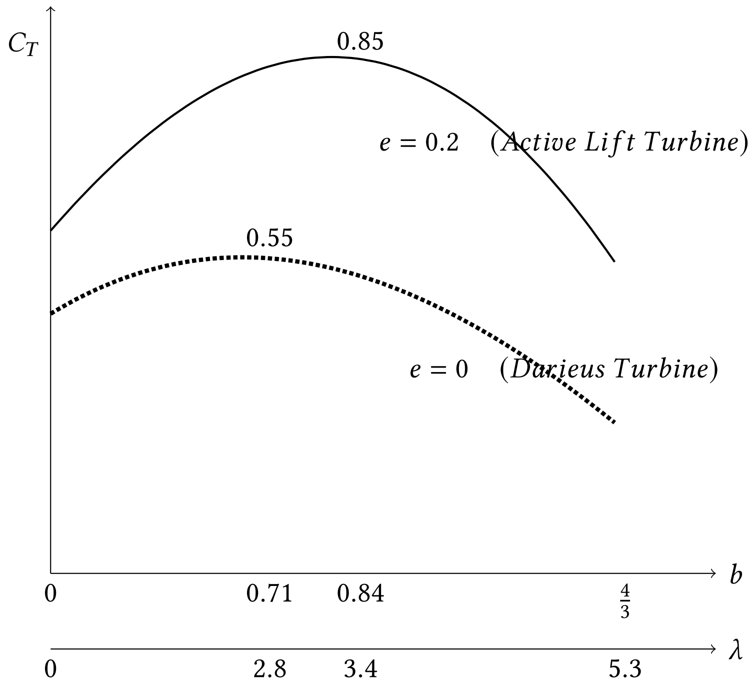

Figure 10: Power coefficient for HAWT and ALWT

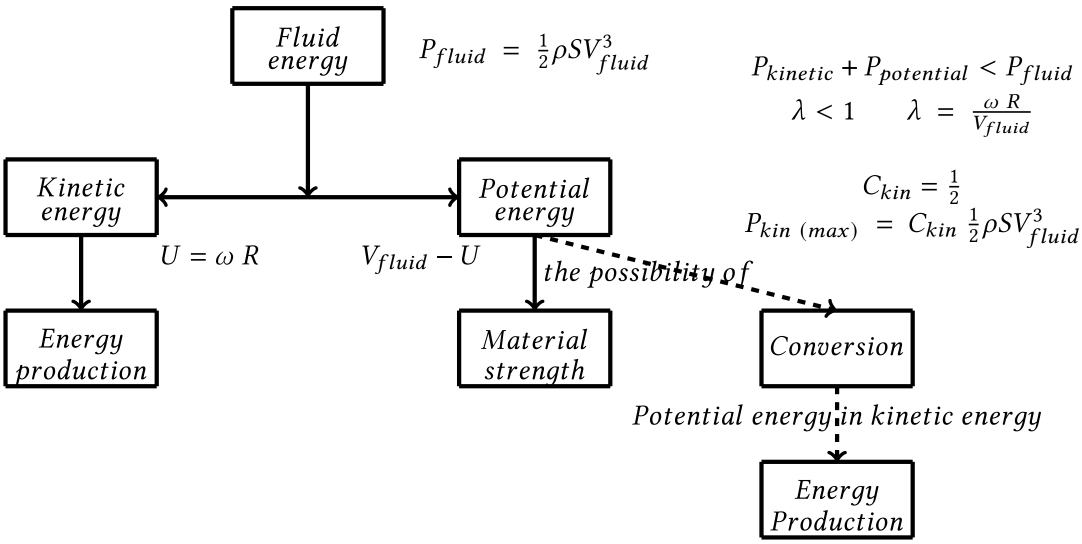

Figure 11: Energy schematization for slow-speed turbines

$$\left[\,1 + \frac{16}{27}\,\right] \geq C_{Total\ maxi\ \lambda>1} \geq \frac{16}{27}$$

Using piezo-electric materials would make it possible to transform potential energy into electrical energy (Piezo-electric materials efficiency is currently very low). However, in order to achieve higher efficiency, the potential energy should be transformed into kinetic energy. Thus, the power coefficient will be $C_{total} = C_{kin} + C_{pot}$ whatever the type of turbine. Compared to a HAWT turbine, the gain of a VAWT Turbine with an energy recovery system should be from 20% to 50%. Concerning a vertical axis turbine with a conversion system, the power factor is higher than the one defined by Betz. In the comparative table, a yield of 0.6 was chosen for the mechanical conversion system of the potential energy into mechanical energy. By choosing an efficient technology, this yield could be greatly increased, thus improving the turbine performance. The definition of the maximum power coefficient is the one established by Betz which remains valid for horizontal axis turbines HAWT but not for vertical axis turbines VAWT. The given results examined a conversion of the potential energy into kinetic energy through a mechanical system which is not applicable for horizontal axis turbines HAWT. The calculation of the powers are the sum of the powers taken into account.

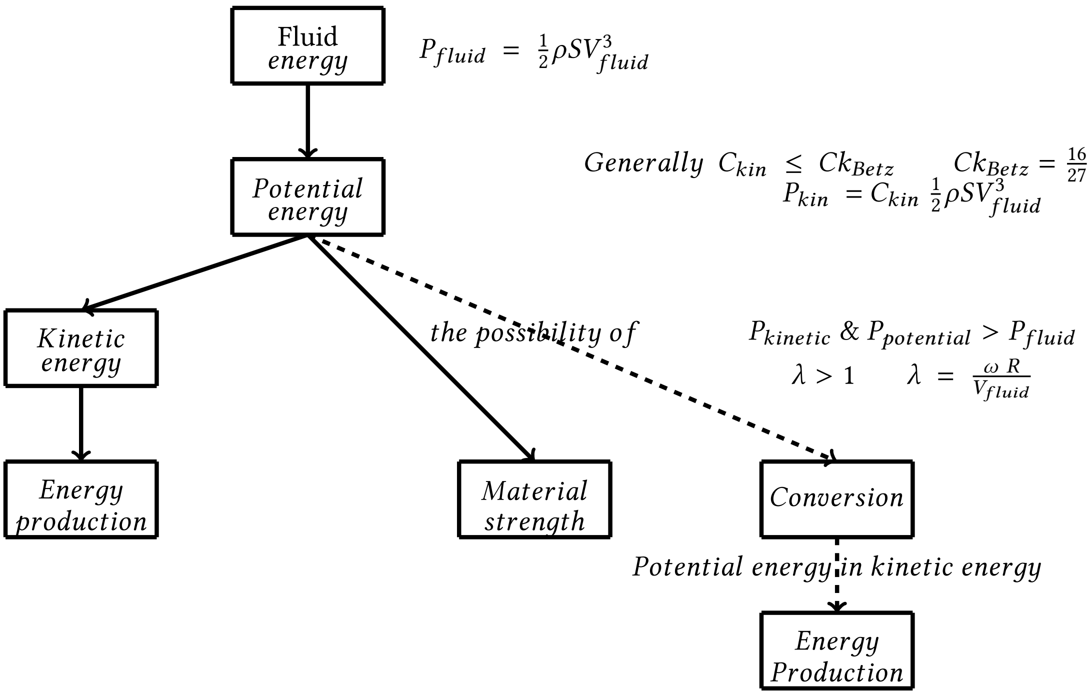

Figure 12: Energy diagram for high-speed wind turbines

8 Conclusion

The paper presents a new formulation for power calculation of wind or water turbines. Taking into account the potential energy allows to establish that the conservation of energy is well verified. This was not the case in the formulation presented by Betz. Until now, potential energy has never been taken into account when defining turbine power coefficients. A new power coefficient has been defined for the turbines depending on their type (low or high speed). As described in paragraph section 6, a significant increase of yield is possible. Obtaining much higher gains will require research and technical ingenuity currently, a study is underway for a wind turbine with a higher efficiency.

A Appendices:

A.1 Energy conservation

(Prescott Joule 1884) contributed to the definition of the conservation of energy through his work on the equivalence of work and heat. This was established more precisely by James Prescott Joule through his famous experiments of 1843-1847, in which a falling weight heats up the water in a calorimeter thanks to the friction of a weight-driven impeller in the water. For a fluid in circulation without exchange of energy with the outside, the energy of the fluid can be converted into kinetic energy and potential energy.

$$E_{kin} + E_{pot} = E_{kin\ fluid}$$

Kinetic energy can be a driving energy to drive, for example, an electric generator. In case of variation of the motor resistive torque, the kinetic energy of the fluid is not modified and causes a variation of the kinetic energy of the fluid which results in a variation in the opposite direction of the potential energy (conservation of the energy).
The equation becomes

$$\Rightarrow \qquad \frac{dE_{kin}}{dt} = -\frac{dE_{pot}}{dt} \tag{18}$$

A.2 Calculation of power coefficients depending on the type of turbine

A.2.1 Low speed $\lambda < 1$

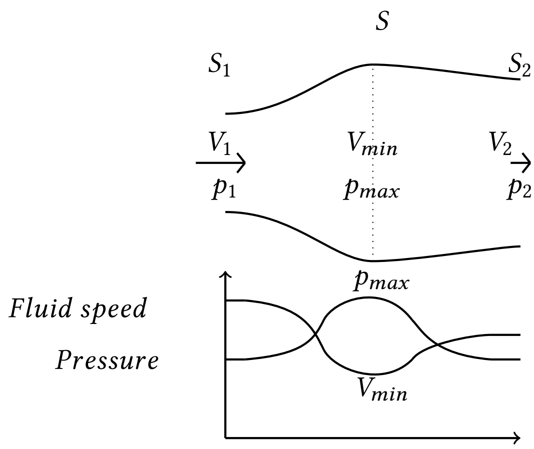

Figure 13: Current tube and evolution of speed and pressure

The figure 13 is a curve for pressure and velocity. The curves may be different for different types of technology, but pressure and velocity are continuous functions. Recoverable kinetic power is limited. There can be no production of kinetic energy without potential energy.

$$E_{kin-max-productive} \le min(\ E_{kin}\ ,\ E_{pot}\ ) \tag{19}$$

If the variation in kinetic energy increases, the variation in potential energy decreases and vice versa.

$$\frac{dE_{kin}}{dt} \nearrow \frac{dE_{pot}}{dt} \searrow \qquad or \qquad \frac{dE_{kin}}{dt} \searrow \frac{dE_{pot}}{dt} \nearrow \tag{20}$$

Energy conservation requires

$$\frac{dE_{kin}}{dt} = -\frac{dE_{pot}}{dt}$$

In accordance with conditions equation (19) and 20 and that the energy of the fluid is converted into potential energy and kinetic energy (equation (7)), the optimum is in the case of $E_{kin} = E_{pot}$
The kinetic power of the turbine is defined by this formula

$$P_{kin} = C_{kin}\ \frac{1}{2}\ \rho S V_{fluid}^{3} \qquad C_{kin} = 4\ a^2\ (1-a)$$

The possible values of a are for the solution of the equation $4\ a^2\ (1-a) = \frac{1}{2}$

$$a = 0.8090167 \qquad a = \frac{1}{2} \qquad a = -0.3090167$$

a must be positive and in the case $a = \frac{1}{2}$, V_2 would be zero and S_2 infinite.
The only possible solution is

$$a=0.8090167 \tag{21}$$

$$P_{kin} = C_{kin}\,\frac{1}{2}\,\rho S V_{fluid}^3 \qquad C_{kin} \leq \frac{1}{2} \tag{22}$$

A.2.2 High speed turbine $\lambda > 1$

The pressure difference across the turbine and the variation in fluid speed can be schematized as follows: figure 14. For high speed turbines, an increase in wind speed results in a variation in potential energy and

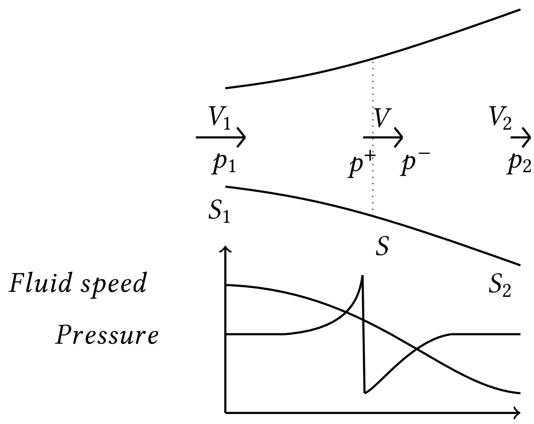

Figure 14: Current tube and evolution of speed and pressure

kinetic energy. This is consistent with energy conversation, as it introduces the notion of temporal energy transfer (equation (10)).

$$\frac{dE_{kwind}(t)}{dt} > 0 \quad \Rightarrow \quad \frac{dE_{pot}(t+\Delta\, t)}{dt} > 0 \quad \Rightarrow \quad \frac{dE_{kin}(t+2\Delta\, t)}{dt} > 0$$

As there cannot be production of kinetic energy without potential energy (equation (19) $E_{kin-max-productive} \leq min(\ E_{kin}\ ,\ E_{pot}\ )$), this implies :

$$E_{pot}(t+\Delta\, t) \ = \ E_{kin}(t+2\Delta\, t)$$

Kinetic power can be determined from kinetic energy and its expression is

$$P_{kin} = \ C_{kin}\,\frac{1}{2}\,\rho S V_{fluid}^3 \qquad\qquad C_{kin} \ = \ 4\,a^2\,(1-a) \tag{23}$$

Searching for the maximum kinetic power (Betz limit) :

$$\frac{dP_{kin}}{dt} = 0 \qquad a(2-3a) = 0 \qquad as \qquad a \geq \frac{1}{2} \qquad a = \frac{2}{3} \qquad C_{kin-maxi} = \frac{16}{27} = C_{kin-Betz} \tag{24}$$

For $a = \frac{2}{3}$

$$P_{kin} \ = \ C_{kin}\,\frac{1}{2}\,\rho\;S\;V_{fluid}^3 \ = \ \frac{C_{kin}}{a}P_{fluid} \ = \ \frac{8}{9}\,P_{fluid}$$

$$C_{kin} \ = \ C_{pot} \ = \ 4 * a^2(1-a) \ = \ \frac{16}{27} \ \approx 60\% \tag{25}$$

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